

# Magnetic Field Effect on the Phase Transition in AdS Soliton Spacetime

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## Abstract

We investigate the scalar perturbations in an anti-de Sitter soliton background coupled to a Maxwell field via marginally stable modes. In the probe limit, we study the magnetic field effect on the holographic insulator/superconductor phase transition numerically and analytically. The condensate will be localized in a finite circular region for any finite constant magnetic field. Near the critical point, we find that there exists a simple relation among the critical chemical potential, magnetic field, the charge and mass of the scalar field. This relation indicates that the presence of the magnetic field causes the transition from insulator to superconductor to be difficult.

## 1 Introduction

The AdS/CFT correspondence [1] provides a powerful theoretical method to understand the strongly coupled field theories in condensed matter physics. A holographic superconductor (superfluid) model has been constructed recently in the work [2, 3](for reviews

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see [4]). The dual gravitational configurations are some anti-de Sitter (AdS) black holes with/without some charged matter contents.

The phase transition studied in Refs.[2, 3] is actually a holographic superconductor/metal phase transition. The holographic superconductor model can be simply constructed by an Einstein-Maxwell theory with a negative cosmological constant coupled to a complex scalar field. In particular, when the temperature of the black hole is below a critical temperature, there are at least two distinct mechanisms leading to the black hole solution unstable to develop a scalar hair near the horizon [5]. And the condensation of the scalar hair induces the local U(1) symmetry breaking of the system, which gives the nonvanishing vacuum expectation value to the dual charged operator in the boundary field theory. Therefore, the U(1) symmetry breaking in gravity leads to a breaking of the global U(1) symmetry in the dual field theory. This results in a superconductor (superfluid) phase transition.

The holographic insulator/superconductor phase transition was first researched in Ref.[6]. A five-dimensional AdS soliton background [7] coupled to a Maxwell field and a scalar field was used to model the holographic insulator/superconductor phase transition at zero temperature. The normal phase in the AdS soliton is dual to a confined gauge theory with a mass gap which is reminiscent of an insulator phase [8]. As the chemical potential grows sufficiently up to a critical value, the instability is triggered, resulting in the emergence of the scalar hair which is dual to a superconducting phase in the boundary field theory. The holographic insulator/superconductor phase transition was also investigated in Refs.[9, 10, 11, 12, 13, 14, 15].

There are some discussions of holographic superconductors in the presence of magnetic field [5, 16, 17, 18, 19, 20, 21], but they mainly focused on superconductor/metal phase transition. Motivated by these studies and especially by our previous works [13, 22], we expect to explore how the magnetic field impacts the behavior of the condensate if a magnetic field is added into the AdS soliton spacetime. We are working in the probe limit, which means that the scalar field and Maxwell field have no back-reaction to the gravity background.

In this paper, we first adopt the idea of marginally stable modes [2] to study the holographic insulator/superconductor phase transition in AdS soliton background at zero temperature. We employ the method introduced in Ref.[23] to study the quasinormal modes(QNMs) of the scalar perturbations of the system just like the one in Ref.[6]. At some critical values of the chemical potential  $\mu$  and magnetic field  $B$ , the marginally stable modes will emerge. This represents that at the critical parameters, the AdS soliton background becomes unstable and will be fond of an AdS soliton background coupled with the hair of charged scalar fields. In particular, for some given mass  $m^2$  and charge  $q$ , the special combination of the critical chemical potential  $\mu$  and the critical magnetic field  $B$  satisfies a simple relation, *i.e.*  $q^2\mu^2 - |qB| = \Lambda^2$ , where  $\Lambda$  is a positive constant. The minus

sign in the combination is interesting because, as the magnetic field grows stronger, the critical chemical potential becomes higher. This indicates that the presence of the magnetic field causes the transition from insulator to superconductor to be difficult. Setting the magnetic field vanishing, the critical chemical potentials we derived are consistent with the results obtained by previous works [13, 22]. Actually, corresponding to various critical parameters, there are multiple marginally stable modes related to the nodes  $n = 1, 2, 3 \dots$ , which are unstable due to the oscillations of the field in the radial direction [2, 24]. Taking advantage of the shooting method, we plot the profile of the scalar field depending on the radial direction. From these diagrams, one can intuitively see the “nodes” of the scalar field. Furthermore, using the variational method for the Sturm-Liouville eigenvalue problem [25], we analytically study the holographic insulator/superconductors phase transition, following the previous work in Ref.[13]. Near the critical point, we show a simple relation among the critical chemical potential, magnetic field, the charge and mass of the scalar field.

The paper is organized as follows: in Sec.2, we introduce the AdS soliton background and obtain the equations of motion in the probe limit. We study the marginally stable modes in Sec.3. In Sec.4 the system is solved by shooting method. In Sec.5 we extract the relation of the four parameters at the critical phase transition points using the variational method for the Sturm-Liouville eigenvalue problem. Conclusions and discussions are drawn in Sec.6.

## 2 The Background

We construct the model of the holographic insulator/superconductor phase transition with the Einstein-Maxwell-scalar action in five-dimensional spacetime:

$$S = \int d^5x \sqrt{-g} (R + \frac{12}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |\nabla_\mu \psi - iq A_\mu \psi|^2 - m^2 |\psi|^2), \quad (1)$$

where  $L$  is the radius of AdS spacetime. When the Maxwell field and scalar field are absent, the above action admits the AdS soliton solution [7] :

$$ds^2 = L^2 \frac{dr^2}{f(r)} + r^2 (-dt^2 + d\rho^2 + \rho^2 d\theta^2) + f(r) d\chi^2. \quad (2)$$

where  $f(r) = r^2 - r_0^4/r^2$ .<sup>1</sup> The asymptotical geometry approaches to  $R^{1,2} \times S^1$  near the boundary. Moreover, the Scherk-Schwarz compactification  $\chi \sim \chi + \pi L/r_0$  is needed to obtain a smooth geometry. This gives a dual picture of a three-dimensional field theory with a mass gap, which resembles an insulator in the condensed matter physics. The geometry in  $(r, \chi)$  directions just looks like a cigar whose tip is at  $r = r_0$ . The temperature in this background is zero.

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<sup>1</sup> We will work in polar coordinates  $dx^2 + dy^2 = d\rho^2 + \rho^2 d\theta^2$  in this paper.

The equations of motion (EoMs) of matter fields are

$$(\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)\psi - m^2\psi = 0, \quad (3)$$

$$\nabla_\nu F^{\nu\mu} = iq[\bar{\psi}(\nabla^\mu - iqA^\mu)\psi - \psi(\nabla^\mu + iqA^\mu)\bar{\psi}]. \quad (4)$$

The boundary conditions for the matter fields near the infinity  $r \rightarrow \infty$  of the AdS soliton are

$$\psi = \psi^{(1)}r^{-2+\sqrt{4+m^2}} + \psi^{(2)}r^{-2-\sqrt{4+m^2}} + \dots, \quad (5)$$

$$A_t = \mu - \frac{\rho_e}{r^2} + \dots, \quad (6)$$

where  $\psi^{(i)} = \langle \hat{O}_{(i)} \rangle$ ,  $i = 1, 2$ , and  $\hat{O}_{(i)}$  are the corresponding dual operators of  $\psi^{(i)}$  in the boundary field theory. The conformal dimensions of the operators are  $\Delta_\pm = 2 \pm \sqrt{4+m^2}$ .  $\mu$  and  $\rho_e$  are the corresponding chemical potential and charge density in the boundary field theory.

Following Ref.[6], we work in the probe limit, where the Maxwell field and scalar field do not back react on the background metric. When there is no condensate, *i.e.*  $\psi = 0$ , EoMs have the solution

$$A = \mu dt + \frac{1}{2}B\rho^2 d\theta. \quad (7)$$

In addition to a constant chemical potential  $\mu$ , adding a constant magnetic field  $B$  to the Maxwell field is also consistent in the probe limit.

As the scalar field  $\psi$  does not vanish, we must solve the coupled EoMs. However, near the critical point of phase transition the scalar field is nearly zero, we can treat  $\psi$  as a probe into this background which is a neutral AdS soliton with a constant electric potential and magnetic field. We are interested in axisymmetric solutions in which all fields are independent of  $\theta$ . We consider an Ansatz of the form  $\psi = F(t, r)H(\chi)U(\rho)$ . Substituting  $\psi$  into Eq.(3) and making the separation of the variables, we can obtain

$$\begin{aligned} & \frac{\partial^2 F(t, r)}{\partial r^2} + \left(\frac{3}{r} + \frac{\partial_r f}{f}\right) \frac{\partial F(t, r)}{\partial r} - \frac{L^2}{fr^2} \frac{\partial^2 F(t, r)}{\partial t^2} + \frac{2iq\mu L^2}{fr^2} \frac{\partial F(t, r)}{\partial t} \\ & + \frac{L^2}{fr^2} (q^2\mu^2 - m^2r^2 - \frac{\lambda^2 r^2}{f} - k^2) F(t, r) = 0, \end{aligned} \quad (8)$$

where  $\lambda^2$  and  $k^2$  are the eigenvalues of the following equations, respectively:

$$\frac{d^2 H(\chi)}{d\chi^2} = -\lambda^2 H(\chi), \quad (9)$$

$$\frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{d}{d\rho} U(\rho) \right] - \frac{1}{4} q^2 B^2 \rho^2 U(\rho) = -k^2 U(\rho), \quad (10)$$

where  $\lambda = 2r_0 l/L$ ,  $l \in \mathbb{Z}$  owing to the periodicity of  $H(\chi) = H(\chi + \pi L/r_0)$ .  $U(\rho)$  solves the equation for a two-dimensional harmonic oscillator with frequency determined by  $B$  and  $k^2 = n|qB|$ ,  $n \in \mathbb{Z}^+$ . We expect that the lowest mode  $l = 0, n = 1$  will be the first to condense and result in the most stable solution after condensing. We can also set  $L = 1$  and  $r_0 = 1$  without loss of generality. We finally obtain the equation of motion of  $F(t, r)$ :

$$\begin{aligned} & \frac{\partial^2 F(t, r)}{\partial r^2} + \left(\frac{3}{r} + \frac{\partial_r f}{f}\right) \frac{\partial F(t, r)}{\partial r} - \frac{L^2}{fr^2} \frac{\partial^2 F(t, r)}{\partial t^2} + \frac{2iq\mu L^2}{fr^2} \frac{\partial F(t, r)}{\partial t} \\ & + \frac{L^2}{fr^2} (q^2 \mu^2 - |qB| - m^2 r^2) F(t, r) = 0. \end{aligned} \quad (11)$$

Note that in this case,  $U(\rho) = \exp(\frac{-|qB|\rho^2}{4})$ , so for any finite magnetic field, the superconducting condensate will be localized to a finite circular region. As the magnetic field becomes smaller, the region grows until it occupies the whole plane, which can be seen in the profile of  $U(\rho)$  by setting  $B \rightarrow 0$ . We can also see similar phenomena in other gravity background [5, 17].

### 3 Critical behavior via Quasinormal Modes

To reveal the stability of a spacetime background, an effective method is to analyze the QNMs of the perturbations in the fixed background (for reviews, see Refs.[26, 27, 28]. The temporal part of the QNMs behave like  $e^{-i\omega t}$ . Therefore, if the imaginary part of the QNMs is negative, the mode will decay in time and the perturbation will ultimately fade away, indicating that the background is stable against this perturbation. In sharp contrast, if the imaginary part is positive, the background is unstable. The critical case is that if the perturbation has a marginally stable mode, *i.e.*  $\omega = 0$ , one always expects that this is a signal of instability, where a phase transition may occur [2].

In order to study the phase transitions in the background, we further define  $F(t, r) = e^{-i\omega t} R(r)$ , and Eq.(11) becomes

$$R''(r) + \left(\frac{f'}{f} + \frac{3}{r}\right) R'(r) + \frac{1}{fr^2} [(\omega + q\mu)^2 - |qB| - m^2 r^2] R(r) = 0, \quad (12)$$

where a prime denotes the derivative with respect to  $r$ .

Taking advantage of the Horowitz and Hubeny's method [23] to study these QNMs, we find it is convenient to work in the  $z$ -coordinate where  $z = 1/r$ . In this new coordinate, the infinite boundary is now at  $z = 0$ , while the tip is at  $z = z_0 = 1/r_0 = 1$ . Equation (12) becomes

$$R''(z) + \left[-\frac{1}{z} + \frac{f'(z)}{f(z)}\right] R'(z) + \frac{1}{z^4 f(z)} [z^2 (\omega + q\mu)^2 - z^2 |qB| - m^2] R(z) = 0, \quad (13)$$

where we make the prime denote the derivative with respect to  $z$  from now on. Following the procedure of Horowitz and Hubeny [23], we multiply  $z^4 f(z)/(z-1)$  to both sides of the above equation, and we obtain

$$S(z)R''(z) + \frac{T(z)}{z-1}R'(z) + \frac{V(z)}{(z-1)^2}R(z) = 0, \quad (14)$$

where the coefficients are given by

$$S(z) = \frac{z^4 f(z)}{z-1}, \quad (15)$$

$$T(z) = -z^3 f(z) + z^4 f'(z), \quad (16)$$

$$V(z) = \left[ z^2(\omega + q\mu)^2 - z^2|qB| - m^2 \right] (z-1). \quad (17)$$

$S(z)$ ,  $T(z)$  and  $V(z)$  are all polynomials and can be expanded to a finite order as

$$S(z) = \sum_{j=0}^J s_j (z-1)^j, \quad (18)$$

where  $J$  is a finite integer.  $T(z)$  and  $V(z)$  can be polynomially expanded similarly.

Because of the absence of a black hole horizon in the AdS soliton, the boundary condition at the tip is a finite quantity. This motivates us to find a solution like

$$R(z) = \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n (z-1)^n. \quad (19)$$

Substituting Eqs.(19) and (18) into Eq.(14) and comparing the coefficients of  $(z-1)^n$  for the same order, we find that

$$a_n = -\frac{1}{H_n} \sum_{k=0}^{n-1} [k(k-1)s_{n-k} + kt_{n-k} + v_{n-k}]a_k, \quad (20)$$

$$H_n = n(n-1)s_0 + nt_0 + v_0 = -4n^2. \quad (21)$$

We set  $a_0 = 1$  for simplicity because of the linearity of Eq.(14). The boundary condition for the scalar field at  $z = 0$  is

$$R(0) = \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n (-1)^n = 0. \quad (22)$$

And the algebraic equation (22) can solve the modes  $\omega$ .

We restrict  $q = 1$  just like the one in Ref.[6] in the numerical calculations. In practice, we will expand  $R(z)$  to a large order which is  $N = 300$  in this paper. Because of the

fact that the real part of the frequency indicates the energy of the mode, we only take care of the QNMs with a positive real part. From the numerical results which are plotted in Figure.(1), we find that when a marginally stable mode arises, the square of chemical potential  $\mu^2$  and magnetic field  $B$  satisfy a simple linear relation, whose slope is particularly one, and the intercept just gives the square of critical chemical potential in the absence of magnetic field, which is consistent with our previous work [22]. For convenience, we define  $\mu^2 - |B| \equiv \Lambda^2$  which determines the stability of the system. The minus sign in the combination is interesting, which means as the magnetic field grows stronger, the critical chemical potential becomes higher. This indicates that the presence of the magnetic field causes the transition from insulator to superconductor to be difficult. By increasing  $\Lambda$  to some critical value, the marginally stable modes will emerge, or rather, a phase transition may occur.

Table 1: The first three lowest-lying  $\Lambda_n$ 's for various mass squares obtained from the calculation of the marginally stable modes.

	$m^2 = -15/4$	$m^2 = -3$	$m^2 = -7/3$	$m^2 = -2$	$m^2 = -1$
$\Lambda_0$	1.8849	2.3963	2.6903	2.8145	3.1346
$\Lambda_1$	4.2263	4.7926	5.1132	5.2456	5.5889
$\Lambda_2$	6.6032	7.1888	7.5172	7.6554	8.0092

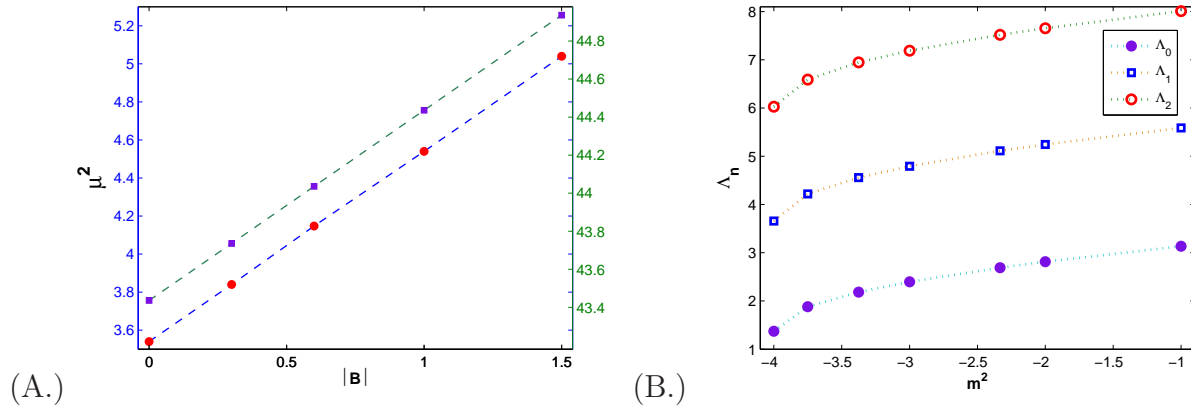


Figure 1: (A.) The square of the critical chemical potentials for the first (red dots) and the third (purple quadrate points) lowest-lying modes versus magnetic field at  $m^2 = -15/4$ . (B.) The first three lowest-lying  $\Lambda_n$ 's for the marginally stable modes versus  $m^2$  of the scalar field. The points are numerically obtained from the QNMs.

Table (1) shows the first three lowest-lying critical  $\Lambda_n$ 's in which the index  $n$  denotes

the “overtone number” for various  $m^2$ ’s. Marginally stable modes corresponding to higher overtone numbers  $n = 1, 2 \dots$  may also appear. However, they are unstable due to the oscillations in the  $z$  direction, which can be explicitly seen in the next section. The nodes  $n = 0, 1, 2$  can also be intuitively seen in the next section. The part B of Fig.(1) shows  $\Lambda_n$ ’s of the marginally stable modes for various squared mass of the scalar field. Turning off the magnetic field, we recover the results previously derived in Ref.[22]. Therefore, despite that we do not exactly know the phase structures through the marginally stable modes, they can reveal the onset of the phase transition in practice.

## 4 Critical behavior via the shooting method

Shooting method is an alternative way to study the critical behavior of the phase transition, which has been widely used in the previous studies on holographic superconductors [4]. In this section, we will make use of the shooting method to study the critical behavior in AdS soliton background, especially to plot the profile of the scalar field. We will also compare it with the above quasinormal modes method.

We focus on the static case in which  $F$  is independent of  $t$ ; we can get equation of motion by simply setting  $\omega = 0$  in Eq.(13):

$$R''(z) + \left[-\frac{1}{z} + \frac{f'(z)}{f(z)}\right]R'(z) + \frac{1}{z^4 f(z)}[z^2(q^2\mu^2 - |qB|) - m^2]R(z) = 0. \quad (23)$$

Near the boundary  $R(z)$  behaves as

$$R|_{z \rightarrow 0} = \varphi^{(1)} z^{2-\sqrt{4+m^2}} + \varphi^{(2)} z^{2+\sqrt{4+m^2}} + \dots \quad (24)$$

It is well known that when  $0 < \sqrt{4+m^2} < 1$ , the scalar field admits two different quantizations related by a Legendre transform [29].  $\varphi^{(i)}$  can either be interpreted as a source or an expectation value. In this paper, we will only consider the case where the faster falloff is dual to the expectation value, *i.e.*  $\varphi^{(1)} = 0$ .

We now study the behavior of the solution near the tip  $z = 1$ . Then Eq.(23) becomes

$$R''(z) - \frac{1}{1-z}R'(z) + \frac{\kappa}{4(1-z)}R(z) = 0, \quad (25)$$

where  $\kappa \equiv q^2\mu^2 - |qB| - m^2 \neq 0$ .<sup>2</sup> It has a solution near  $z = 1$  as

$$R|_{z \rightarrow 1} = \alpha + \beta \log(\sqrt{|\kappa|}(1-z)) + \dots, \quad (26)$$

where  $\alpha$  and  $\beta$  are two constants. Since we want the field to be finite at the tip, we choose the boundary condition by setting  $\beta = 0$ .

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<sup>2</sup>Note that we neglect the particular case  $\kappa = 0$ . One can also consider this case, but it does not matter.



To make use of the shooting method, we begin with an initial value of  $R(z)$  at the tip  $z_0$  and then calculate the EoM of  $R(z)$  [Eq.(23)] numerically provided that the infinite boundary condition  $\varphi^{(1)} = 0$  is satisfied. We also take  $q = 1$  for simplicity. We can set  $R(1)$  as an arbitrary constant due to the linearity of Eq.(23). In addition, near the critical point of the phase transition the quantity of  $R(z)$  is very close to zero, therefore, we can impose the following initial conditions at the tip  $z = 1$ :

$$R(1) = 0.001, \quad R'(1) = \frac{\kappa}{4}R(1). \quad (27)$$

For a given  $m^2$ , only for certain values of  $\mu$  and  $B$  do we get to satisfy the boundary conditions.

Note that the chemical potential  $\mu$  and magnetic field  $B$  appear only as a whole  $q^2\mu^2 - |qB|$  in Eq.(23) and especially in  $\kappa$ . And inspired by the result from marginally stable modes that only the combination  $\Lambda^2 \equiv \mu^2 - |B|$  determines the occurrence of instability, we expect that this result may be confirmed by our shooting method. Just as expected, we find that only the appropriate  $\Lambda^2$  can trigger the condensate. Furthermore, the values of  $\Lambda_c$  obtained from the shooting method are perfectly consistent with the  $\Lambda_n$ 's in Table (1) derived from the ‘‘marginally stable modes’’ method. So we will not distinguish between them from now on.

Figure.(2) exhibits the multiple marginally stable curves of the scalar fields for various  $m^2$ . For example, the first three lowest-lying modes in the plot of  $m^2 = -15/4$  are in the sequence  $\Lambda_0 < \Lambda_1 < \Lambda_2$ . The red line is dual to the minimal value of  $\Lambda_c$ , which has no intersecting points with the  $R(z) = 0$  axis at nonvanishing  $z$ . Therefore, we consider the mode corresponding to  $\Lambda_c \approx 1.8816$  as a mode of node  $n = 0$ . Further, we regard the green line dual to  $\Lambda_c \approx 4.2341$  and blue line related to  $\Lambda_c \approx 6.6156$  as modes with nodes  $n = 1$  and  $n = 2$ , respectively. However, the green and blue lines are thought to be unstable due to the fact that the radial oscillations in  $z$ -direction of  $R(z)$  will cost energy [24]. In addition, the above discussions also hold for other diagrams in Fig.(2). It is interesting to note that the marginally stable curves corresponding to different  $\Lambda_c$ 's are just the division of the curves of  $R(z)$  with different nodes.

## 5 Critical behavior via the Sturm-Liouville method

From the above numerical analysis, we can see that when the combination of chemical potential  $\mu$  and magnetic field  $B$ , which is  $\Lambda^2 \equiv \mu^2 - |B|$ , exceeds a critical value  $\Lambda_0^2$  for given mass, the condensations of the operators will turn out. This can be regarded as a superconductor (superfluid) phase. However, when less than  $\Lambda_0$ , the scalar field is vanishing and this can be interpreted as the insulator phase because this system has a mass gap, which is due to the confinement in the (2+1)-dimensional gauge theory via the Scherk-

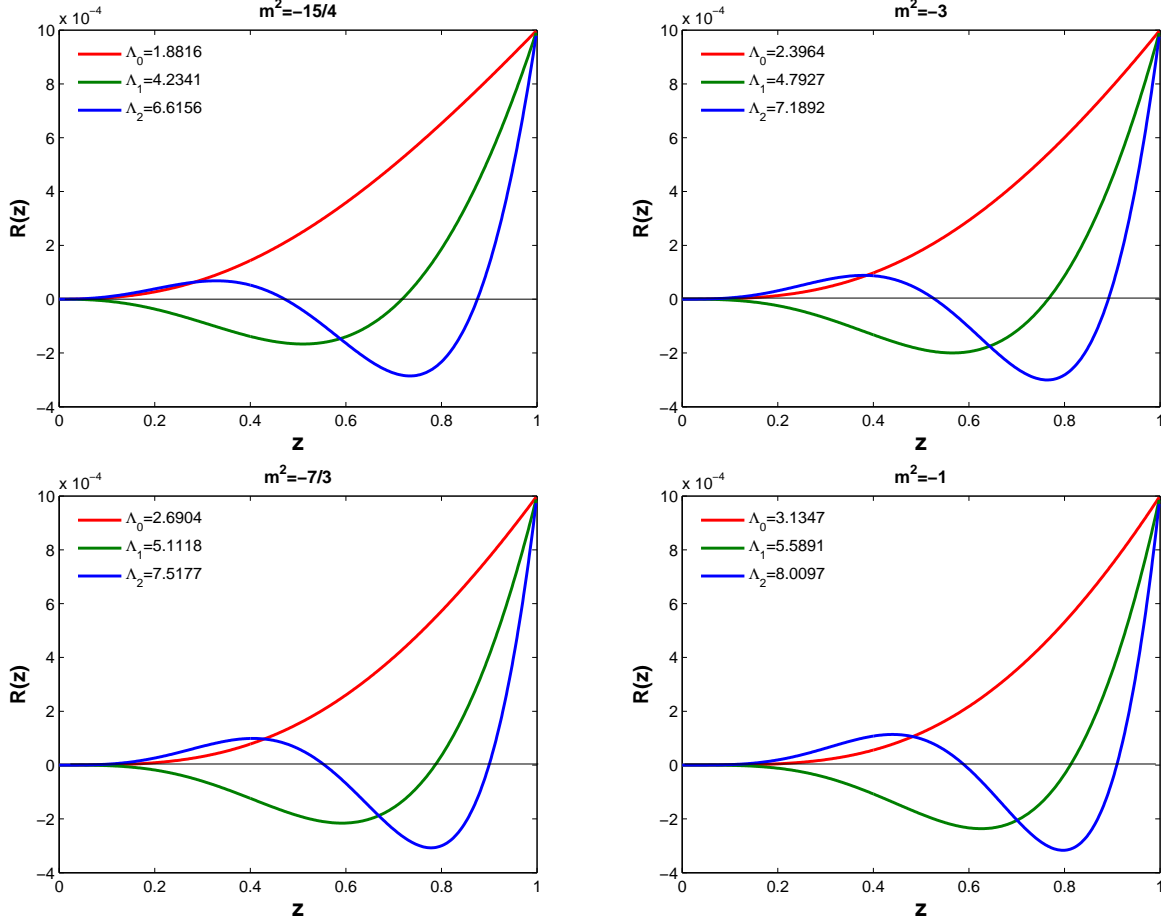


Figure 2: The marginally stable curves of the scalar field corresponding to various critical  $\Lambda_c$ 's in the cases of different mass square. The critical  $\Lambda_c$ 's for different curves are  $\Lambda_0 < \Lambda_1 < \Lambda_2$  in the sequence .

Schwarz compactification. Therefore, the critical parameters satisfied,  $\Lambda_0 = \mu^2 - |B|$ , are the turning points of this holographic insulator/superconductor phase transition.

In this section, using the variational method for the Sturm-Liouville eigenvalue problem [25], we analytically study the phase transition just following the procedure in Ref.[13]. Especially, we look forward to giving an explicit demonstration to our previous numerical results and trying to find an approximate function to relate parameters  $\{q, \mu, B, m^2\}$  at the critical phase transition point. We start from Eq.(23) and also focus on the case  $\varphi^{(1)} = 0$ . The operator  $\hat{O}_{(2)}$  is normalizable when  $m^2 > m_{\text{BF}}^2 = -4$ , where  $m_{\text{BF}}^2$  is the Breitenlohner-Freedman bound of the mass square of scalar field in the AdS spacetime.

Following the steps in Ref.[13], we introduce a trial function  $\Gamma(z)$  into  $R(z)$  near  $z = 0$

as

$$R|_{z \rightarrow 0} \simeq \langle \hat{O}_{(2)} \rangle z^{2+\sqrt{4+m^2}} \Gamma(z). \quad (28)$$

The boundary conditions for  $\Gamma(z)$  are  $\Gamma(0) = 1$  and  $\Gamma'(0) = 0$ . It is easy to obtain the EoM of  $\Gamma(z)$  as

$$\begin{aligned} \Gamma''(z) + \frac{z^4(5 + 2\sqrt{4+m^2}) - 1 - 2\sqrt{4+m^2}}{z(z^4 - 1)} \Gamma'(z) + \frac{|qB| - q^2\mu^2}{z^4 - 1} \Gamma(z) \\ + \frac{z^2(8 + m^2 + 4\sqrt{4+m^2})}{z^4 - 1} \Gamma(z) = 0. \end{aligned} \quad (29)$$

The EoM of  $\Gamma(z)$  can be rewritten as

$$\begin{aligned} \frac{d}{dz} \left[ \underbrace{z^{1+2\sqrt{4+m^2}}(z^4 - 1)}_K \Gamma'(z) \right] + \underbrace{z^{3+2\sqrt{4+m^2}}(8 + m^2 + 4\sqrt{4+m^2})}_{-P} \Gamma(z) \\ - \underbrace{z^{1+2\sqrt{4+m^2}}}_{Q} (q^2\mu^2 - |qB|) \Gamma(z) = 0. \end{aligned} \quad (30)$$

The minimum eigenvalue of  $q^2\mu^2 - |qB|$  can be obtained by varying the following functional:

$$q^2\mu^2 - |qB| = \frac{\int_0^1 dz (K\Gamma'^2 + P\Gamma^2)}{\int_0^1 dz Q\Gamma^2} \equiv \xi(\gamma, m^2). \quad (31)$$

In order to estimate it, we try to set  $\Gamma(z) = 1 - \gamma z^2$ . Thus, we obtain a particular function  $q^2\mu^2 - |qB| = \xi(\gamma, m^2)$ , from which we can get the minimal value of  $q^2\mu^2 - |qB|$  for given mass. It will be subtle if the function  $\xi(\gamma, m^2)$  has more than one minimum. Fortunately, we can see from part A of Fig.(3) that there indeed exists only one minimum for given mass square. We denote the minimum for given mass square as  $\Lambda[m^2]$ .

Table 2: The critical parameters  $q^2\mu^2 - |qB|$  for various mass squares obtained from the QNMs, shooting method, and S-L method.

	$m^2 = -15/4$	$m^2 = -3$	$m^2 = -7/3$	$m^2 = -1$
QNMs	1.8849	2.3963	2.6903	3.1346
Shooting	1.8816	2.3964	2.6904	3.1347
S - L	1.8904	2.3986	2.6928	3.1373

In Table (2), we list the critical parameters  $\Lambda_0$  for various mass squares obtained from the three methods, *i.e.* the calculation of the marginally stable modes, the shooting

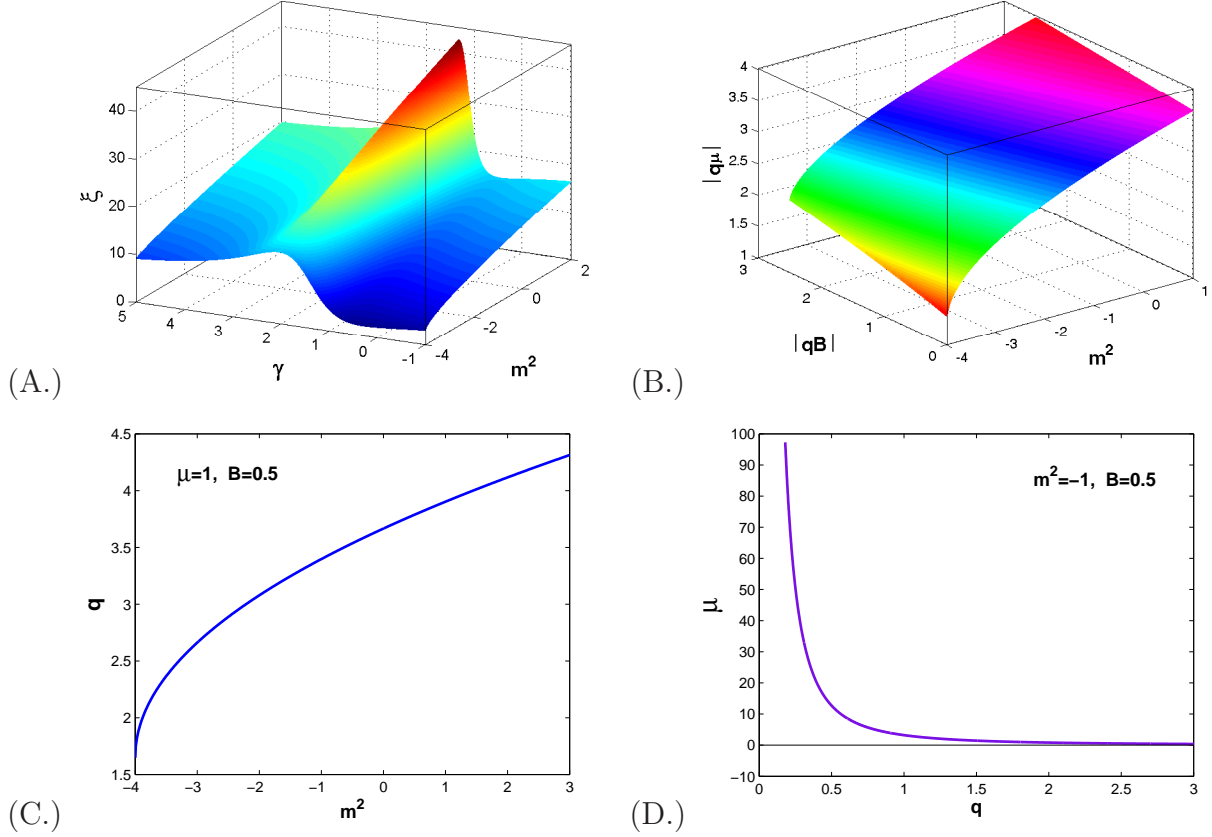


Figure 3: (A.)  $\xi$  as a function of  $\gamma$  and  $m^2$ . (B.) The critical chemical potential  $|q\mu|$  versus  $|qB|$  and  $m^2$ . (C.) The charge  $q$  versus mass square  $m^2$  at fixed chemical potential  $\mu = 1$  and magnetic field  $B = 0.5$ . (D.) The critical chemical potential  $\mu$  versus the charge  $q$  at fixed mass square  $m^2 = -1$  and magnetic field  $B = 0.5$ .

method, and the Sturm-Liouville (S-L) method. The analytical results are in good agreement with numerical values. So we can trust the analytical treatment, especially the choice of  $\Gamma(z) = 1 - \gamma z^2$ .

We finally obtain the simple relation for parameters  $q, \mu, B, m^2$  at the critical phase transition point as

$$q^2 \mu^2 - |qB| = \Lambda[m^2]. \quad (32)$$

This simple relation can explain the main results we obtained. We plot the function  $|q\mu| = |q\mu|(|qB|, m^2)$  in part B of Fig.(3). The relation can be regarded as a function of three arbitrary independent variables in  $\{q, \mu, B, m^2\}$ . We extract the relation between the charge  $q$  and mass square  $m^2$  for the systems whose phase transitions take place at fixed chemical potential  $\mu$  and magnetic field  $B$ . Part C of Fig.(3) shows schematically that when the mass square of the scalar field grows, the charge must grow too. Similarly,

it can be seen from part D of Fig.(3) that the critical chemical potentials are sufficiently depressed by the charge of the scalar field for fixed mass and magnetic field.

## 6 Discussions and Conclusions

In the probe limit, we studied the holographic insulator/superconductor transition in an external constant magnetic field both numerically and analytically. When the magnetic field is absent, the condensate fills the plane homogeneously. On the other hand, noting that the profile of scalar field  $\psi(z, \rho) = R(z) \exp(\frac{-|qB|\rho^2}{4})$  where the profile of  $R(z)$  in superconducting phase is shown in Fig.(2), we see that for any nonvanishing magnetic field, the superconducting condensate will be localized to a finite circular region. As the magnetic field becomes smaller the region grows until it occupies the whole plane, which can be seen in profile of  $U(\rho)$  by setting  $B \rightarrow 0$ .

Despite that we do not exactly know the phase structure by investigating the marginally stable modes of the scalar field, they can actually indicate the onset of the phase transition, which means that the neutral AdS soliton will become unstable to develop charged scalar hairs in this AdS soliton background when the parameters are beyond the critical values. Our results show that only the parameter  $\Lambda^2 = q^2\mu^2 - |qB|$  triggers the instability. Making use of the shooting method to numerically plot the behavior of the scalar field in the radial direction, one can intuitively see the nodes of the marginally stable modes. Furthermore, taking advantage of analytical approach we directly obtained a simple relation which constrains parameters  $\{q, \mu, B, m^2\}$  at the critical point, *i.e.*  $q^2\mu^2 - |qB| = \Lambda[m^2]$ . This relation is the main result of our paper, and it can be used to explain many properties of the model.

Note that in this paper, we have limited ourselves to the choice of  $\psi^{(1)} = 0$  and  $\lambda = 0$ , and  $n = 1$ . It is interesting to study the cases with vanishing  $\psi^{(2)}$  and other  $(\lambda, n)$ 's. Of course, it is also quite significant to investigate the back reaction of matter fields and to draw a full phase diagram of the holographic insulator/superconductor transition. These discussions in the paper are desired to extend to the background of modified gravity, such as including Chern-Simons term and  $R^2$  term[30, 31]. We expect to report on further studies on these and relevant issues.

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